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TECHNISCHE UNIVERSITÄT CHEMNITZ

Technische Universität Berlin

Slicing of Radial Functions: a Dimension Walk in the Fourier Space

Nicolaj Rux | TU Chemnitz | Mathematical Signal and Image Analysis, TUM Science & Study Center Raitenhaslach, 13 March 2025
Joint work with Michael Quellmalz and Gabriele Steidl

The Problem

- We want: **Fast** computation of the kernel sums

$$s_m = \sum_{n=1}^{N} w_n F(\|x_n - y_m\|) \qquad \forall\, m = 1, ..., M$$

- Given: RBF $F\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$, points $x_n, y_m \in \mathbb{R}^d$ and coefficients $w_n \in \mathbb{R}$ for $n = 1, ..., N$, $m = 1, ..., M$
- Goal: Improve computational complexity $\mathcal{O}(MN)$

Examples	$F(\|x\|)$
Gauss	$\exp(-\frac{\|x\|^2}{2\sigma^2})$
Matérn	$\frac{2^{1-\nu}}{\Gamma(\nu)}(\frac{\sqrt{2\nu}}{\beta}x)^\nu K_\nu(\frac{\sqrt{2\nu}}{\beta}x)$
Laplace	$\exp(-\alpha\|x\|)$
Riesz	$-\|x\|^r,\ r \in (0,2)$

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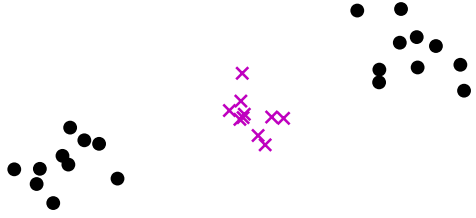
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Motivational Example: Particle Flows

- Given: $\boldsymbol{y} = (y_1, ..., y_M) \in (\mathbb{R}^d)^M$
- Find: $\boldsymbol{x} = (x_1, ..., x_N) \in (\mathbb{R}^d)^N$ that "well approximate" $\boldsymbol{y}$

$$\underset{\boldsymbol{x}}{\operatorname{argmin}} \underbrace{\frac{1}{2N^2} \sum_{m=1}^{N} \sum_{n=1}^{N} F(\|x_n - x_m\|)}_{\text{repulsion}} - \underbrace{\frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} F(\|x_n - y_m\|)}_{\text{attraction}}$$

- Gradient descent approximation:

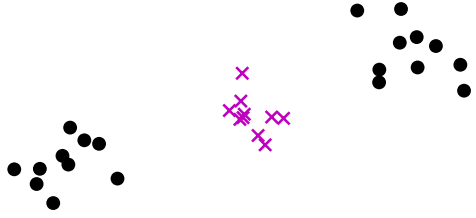


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Content

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   - Motivational Example

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   - Fast Fourier Summation
   - The Slicing formula

3. Computing the Sliced Kernel: A Dimension Walk in Fourier Space
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   - A Fourier inversion formula
   - Tempered Distributions

Fast Fourier Summation

[Kunis Potts 2006] [Potts Steidl Nieslony 2004]

- Fourier expansion of the kernel with degree $N_{\mathrm{ft}} \in \mathbb{N}$

$$F(\|x\|) \approx \sum_{k \in \{-N_{\mathrm{ft}}, \dots, N_{\mathrm{ft}}\}^d} c_k \mathrm{e}^{2\pi \mathrm{i} \langle x, k \rangle}$$

- Insert into the sum (2)

$$s_m = \sum_{n=1}^{N} w_n F(\|x_n - y_m\|) \approx \sum_{n=1}^{N} w_n \sum_{k \in \{-N_{\mathrm{ft}}, \dots, N_{\mathrm{ft}}\}^d} c_k \mathrm{e}^{2\pi \mathrm{i} \langle x_n - y_m, k \rangle}$$

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1. Compute inner sum $\hat{w}_k$ via NFFT of $w \rightsquigarrow \mathcal{O}(N + N_{\mathrm{ft}}^d \log N_{\mathrm{ft}})$
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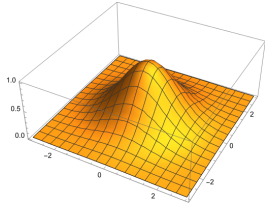
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Slicing Summation

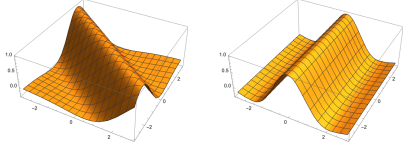
[Hertrich 2024]

- Idea: Construct even $f\colon \mathbb{R} \to \mathbb{R}$ s.t.

$$F(\|x\|) = \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} f(|\langle x, \xi \rangle|)\, \mathrm{d}\xi \qquad \forall\, x \in \mathbb{R}^d$$



radial function $F(\|x\|)$
(Gaussian)



$x \mapsto f(\langle x, \xi \rangle)$ for 2 different ξ

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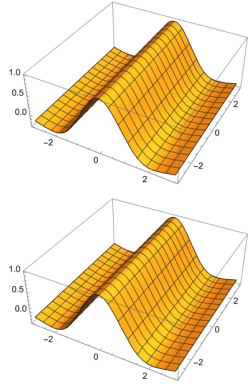
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Choose points $\xi_1, \dots, \xi_P \in \mathbb{S}^{d-1}$

$$\approx \frac{1}{P} \sum_{p=1}^{P} \underbrace{\sum_{n=1}^{N} w_n f(|\langle x_n, \xi_p \rangle - \langle y_m, \xi_p \rangle|)}_{\text{1D kernel sum}}$$

- P one-dimensional sums $\rightsquigarrow \mathcal{O}(P(N + M + N_{\mathrm{ft}} \log N_{\mathrm{ft}}))$



Top: Approximation of F
Bottom: $x \mapsto f(\langle x, \xi \rangle)$

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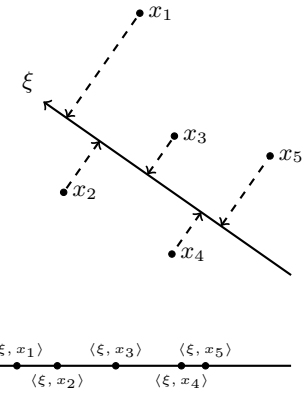
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Projection of points $x_1, \ldots, x_5 \in \mathbb{R}^2$ onto the line in direction ξ

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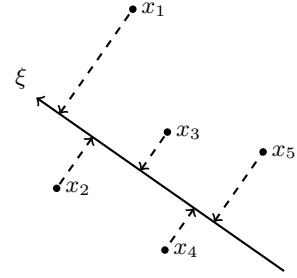
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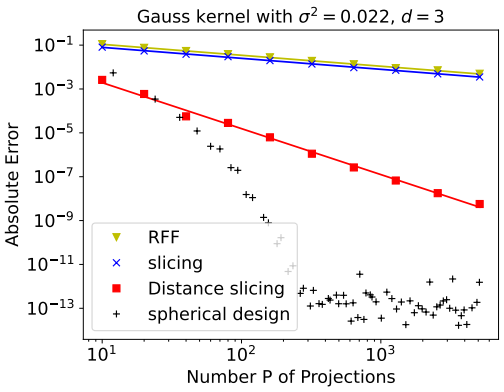
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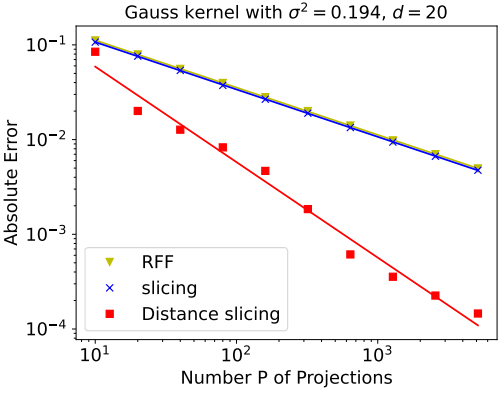


Projection of points $x_1, \dots, x_5 \in \mathbb{R}^2$ onto the line in direction ξ

Slicing Error

[Hertrich Jahn Quellmalz 2025]





$$\left| F(\|x\|) - \frac{1}{P} \sum_{p=1}^{P} f(\langle x, \xi_p \rangle) \right|$$

Content

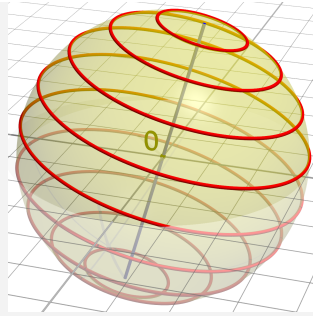
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The Slicing Transform

[M. Quellmalz, R. Beinert, G. Steidl 2023]

Sliced kernel function $f\colon \mathbb{R} \to \mathbb{R}$

$$F(\|x\|) = \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} f(\langle x, \xi \rangle) \, \mathrm{d}\xi \qquad \forall\, x \in \mathbb{R}^d$$



Theorem ($f \mapsto F$) [Hertrich 2024] [Rubin 2003]

For $d \geq 2$, a pair (F, f) of basis functions in $L^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0})$ fulfills this relation if and only if F is the **generalized Riemann–Liouville fractional integral**

$$F(t) = \frac{2\Gamma(\frac{d}{2})}{\sqrt{\pi}\Gamma(\frac{d-1}{2})} \int_0^1 f(ts)(1-s^2)^{\frac{d-3}{2}} \, \mathrm{d}s = \frac{2\Gamma(\frac{d}{2})}{\sqrt{\pi}\Gamma(\frac{d-1}{2})} \frac{1}{t} \int_0^t f(s)(1-s^2/t^2)^{\frac{d-3}{2}} \, \mathrm{d}s.$$

Inverting the Riemann-Liouville fractional integral

$$F(\|x\|) = \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} f(\langle x, \xi \rangle) \, \mathrm{d}\xi \qquad \forall\, x \in \mathbb{R}^d$$

Theorem (Necessary condition for slicing) [R. Quellmalz Steidl 2025]

For $d \in \mathbb{N}$ with $d \geq 3$, let $f \in L^\infty_{\mathrm{loc}}([0,\infty))$, then F is $\lfloor \frac{(d-2)}{2} \rfloor$-times continuously differentiable on $(0,\infty)$.

Theorem (Sufficient condition for slicing) [R. Quellmalz Steidl 2025] [Samko Kilbas Marichev 1993]

For $d \geq 3$, let $F \in \mathcal{C}^{\lfloor d/2 \rfloor}([0,\infty))$ and $G(t) \coloneqq F(\sqrt{t}) t^{(d-2)/2}$, $t \geq 0$. Then F gets sliced by $f \in \mathcal{C}([0,\infty))$ given

$$f(s) \coloneqq \frac{2s}{c_d \Gamma\left((d-1)/2\right)} \left(D_+^{(d-1)/2} G\right)(s^2), \quad s \geq 0,$$

Moreover, if the $\lfloor d/2 \rfloor$-th derivative of F is slowly increasing, then f is slowly increasing.

Inverting the Riemann-Liouville fractional integral

$$F(\|x\|) = \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} f(\langle x, \xi \rangle) \, \mathrm{d}\xi \qquad \forall\, x \in \mathbb{R}^d$$

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Slicing and the Fourier Transform

[R. Quellmalz Steidl 2025]

Assume $F(\|\cdot\|)$ has Fourier transform $\rho(\|\cdot\|)$

$$F(\|x\|) = \mathcal{F}_d[\rho(\|\cdot\|)] = \int_{\mathbb{R}^d} \mathrm{e}^{-2\pi \mathrm{i}\langle x, v\rangle}\, \rho(\|v\|)\, \mathrm{d}v$$

Polar decomposition $v = r\xi$, $r = \|v\|$, $\xi \in \mathbb{S}^{d-1}$, yields

$$= \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathrm{e}^{-2\pi \mathrm{i}\langle x, \xi\rangle r}\, \rho(r) r^{d-1}\, \mathrm{d}r\, \mathrm{d}\xi$$

$$= \frac{1}{2} \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}} \mathrm{e}^{-2\pi \mathrm{i}\langle x, \xi\rangle r} \underbrace{\rho(r)|r|^{d-1}}_{=:\mathcal{M}_d\rho(r)}\, \mathrm{d}r\, \mathrm{d}\xi$$

$$= \frac{1}{2} \int_{\mathbb{S}^{d-1}} \mathcal{F}_1[\mathcal{M}_d\rho](\langle x, \xi\rangle)\, \mathrm{d}\xi$$

$$= \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} \underbrace{\tfrac{|\mathbb{S}^{d-1}|}{2} \mathcal{F}_1[\mathcal{M}_d\rho]}_{=f}(\langle x, \xi\rangle)\, \mathrm{d}\xi$$

We obtained an inversion formula!

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We obtained an inversion formula!

Slicing and the Fourier Transform

[R. Quellmalz Steidl 2025]

Assume $F(\|\cdot\|)$ has Fourier transform $\rho(\|\cdot\|)$

$$F(\|x\|) = \mathcal{F}_d[\rho(\|\cdot\|)] = \int_{\mathbb{R}^d} \mathrm{e}^{-2\pi \mathrm{i}\langle x, v\rangle}\, \rho(\|v\|)\, \mathrm{d}v$$

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Inversion Formula for L^1 Functions

- **Rotation operator** $\mathcal{R}_d$ maps $F\colon [0,\infty) \to \mathbb{R}$ to the radial function

$$\mathcal{R}_d F(x) \coloneqq F(\|x\|), \qquad x \in \mathbb{R}^d$$

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$$\mathcal{A}_d \Phi(r) \coloneqq \tfrac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} \Phi(r\,\xi)\ \mathrm{d}\xi, \qquad r \in \mathbb{R}$$

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Let $d \geq 2$. Let $F\colon [0,\infty) \to \mathbb{R}$ fulfill $\mathcal{R}_d F = \mathcal{F}_d[\mathcal{R}_d \rho]$ for some function $\rho\colon [0,\infty) \to \mathbb{R}$ with $\mathcal{R}_d \rho \in L^1(\mathbb{R}^d)$ and $\mathcal{R}_d F \in L^1(\mathbb{R}^d)$, then

$$f = \tfrac{|\mathbb{S}^{d-1}|}{2} (\mathcal{F}_1 \circ \mathcal{M}_d \circ \mathcal{A}_d \circ \mathcal{F}_d^{-1} \circ \mathcal{R}_d)[F].$$

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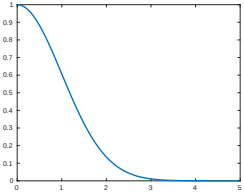
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Example: Gauss Kernel $f = \frac{|\mathbb{S}^{d-1}|}{2}(\mathcal{F}_1 \circ \mathcal{M}_d \circ \mathcal{A}_d \circ \mathcal{F}_d^{-1} \circ \mathcal{R}_d)[F]$

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Figure: Gauss kernel F and the corresponding sliced kernel f for $d = 10$

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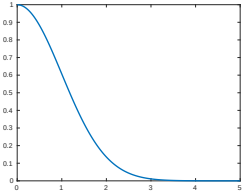


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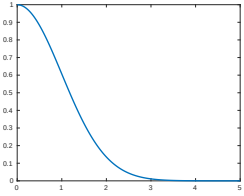


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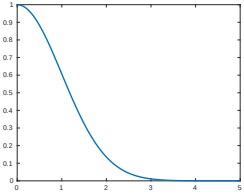
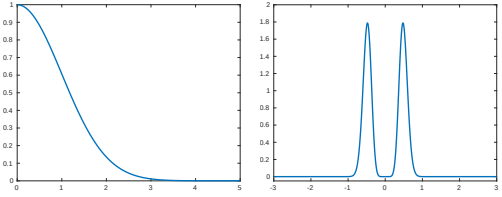


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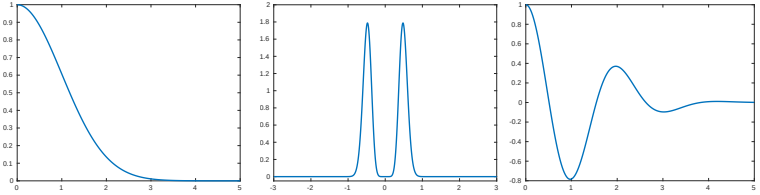


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Slicing of Tempered Distributions

- **Schwartz space** $\mathcal{S}(\mathbb{R}^d) = \left\{\varphi \in C^\infty(\mathbb{R}^d) : D^\alpha x^\beta \varphi(x) \in C_0(\mathbb{R}^d)\, \forall \alpha, \beta \in \mathbb{N}^d\right\}$
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Theorem [R. Quellmalz Steidl 2025]

Let $f \in C(\mathbb{R})$ be slowly increasing and even. Set $F(\|x\|) = \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} f(\langle \xi, x \rangle \, \mathrm{d}\xi$. Then f can be recovered from

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Overview

- Task: compute efficently

$$s_m \coloneqq \sum_{n=1}^{N} w_n F(\|x_n - y_m\|) \quad \forall m = 1, \dots, M$$

- Application in MMD computations
- In one dimension use FFT to reduce complexity $\mathcal{O}(NM) \to \mathcal{O}(N + M + N_{\text{ft}}^d \log(N_{\text{ft}}))$
- In higher dimensions, slice F

$$F(\|x\|) = \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} f(\langle \xi, x \rangle) \, \mathrm{d}\xi \approx \frac{1}{P} \sum_{p=1}^{P} f(\langle \xi_p, x \rangle)$$

- Reduce high dimensional problem to multiple one dimensional problems

$$s_m \approx \sum_{p=1}^{P} \sum_{n=1}^{N} w_n f(\langle \xi, x_n - y_m \rangle$$

- Found inversion formula $f \to F$ in the Fourier sense $f = \frac{|\mathbb{S}^{d-1}|}{2} (\mathcal{F}_1 \circ \mathcal{M}_d \circ \mathcal{A}_d \circ \mathcal{F}_d^{-1} \circ \mathcal{R}_d)[F]$

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Thank you for your attention!

Known Pairs (F, f)

[Hertrich Jahn Quellmalz 2025]

Kernel	$F(x)$	$\mathcal{F}_d^{-1}[F(\|\cdot\|)](\|\omega\|)$	$f(x)$	$\mathcal{F}_1^{-1}[f](\|\omega\|)$
Gauss	$\exp(-\frac{x^2}{2\sigma^2})$	$(2\pi\sigma^2)^{d/2}\exp(-2\pi^2\sigma^2\omega^2)$	${}_1F_1(\frac{d}{2};\frac{1}{2};\frac{-x^2}{2\sigma^2})$	$\frac{\pi\sigma\exp(-2\pi^2\sigma^2\omega^2)(2\pi^2\sigma^2\omega^2)^{(d-1)/2}}{2\Gamma(\frac{d}{2})}$
Laplace	$\exp(-\alpha x)$	$\frac{\Gamma(\frac{d+1}{2})2^d\pi^{\frac{d-1}{2}}}{\alpha^d}(1+\frac{4\pi^2\omega^2}{\alpha^2})^{-\frac{d+1}{2}}$	$\sum_{n=0}^{\infty}\frac{(-\alpha)^n\sqrt{\pi}\Gamma(\frac{n+d}{2})}{n!\Gamma(\frac{d}{2})\Gamma(\frac{n+1}{2})}x^n$	$\frac{\Gamma(\frac{d+1}{2})2^d\pi^{\frac{d-1}{2}}\|\omega\|^{d-1}}{\Gamma(\frac{d}{2})\alpha^d}(1+\frac{4\pi^2\omega^2}{\alpha^2})^{-\frac{d+1}{2}}$
Sliced Laplace	$\frac{I_{\frac{d-2}{2}}(-\alpha t)+\mathbf{L}_{\frac{d-2}{2}}(-\alpha t)}{(\alpha t)^{\frac{d-2}{2}}}$	$\frac{2^{\frac{4-d}{2}}\Gamma(\frac{d}{2})}{\pi^{(d+1)/2}\Gamma(\frac{d-1}{2})\|\omega\|^{d-1}}\frac{2\alpha}{\alpha^2+4\pi^2\omega^2}$	$\frac{2^{\frac{4-d}{2}}}{\sqrt{\pi}\Gamma(\frac{d-1}{2})}\exp(-\alpha x)$	$\frac{2^{\frac{d-4}{2}}}{\sqrt{\pi}\Gamma(\frac{d-1}{2})}\frac{2\alpha}{\alpha^2+4\pi^2\omega^2}$
Matérn	$\frac{2^{1-\nu}}{\Gamma(\nu)}(\frac{\sqrt{2\nu}}{\beta}x)^\nu K_\nu(\frac{\sqrt{2\nu}}{\beta}x)$	$\frac{\Gamma(\frac{2\nu+d}{2})2^{\frac{d}{2}}\pi^{\frac{d}{2}}\beta^d}{\Gamma(\nu)\nu^{\frac{d}{2}}}(1+\frac{2\pi^2\beta^2\omega^2}{\nu})^{-\frac{2\nu+d}{2}}$	[Hertrich 2024, Appx C]	$\frac{\Gamma(\frac{2\nu+d}{2})2^{\frac{d}{2}}\pi^{\frac{d}{2}}\beta^d\|\omega\|^{d-1}}{\Gamma(\frac{d}{2})\Gamma(\nu)\nu^{\frac{d}{2}}}(1+\frac{2\pi^2\beta^2\omega^2}{\nu})^{-\frac{2\nu+d}{2}}$
Riesz for $r>-1$	$-x^r$	not a function	$-\frac{\sqrt{\pi}\Gamma(\frac{d+r}{2})}{\Gamma(\frac{d}{2})\Gamma(\frac{r+1}{2})}x^r$	not a function
Thin Plate Spline	$x^2\log(x)$	not a function	$dx^2\log(x)+C_d x^2$, with $C_d=\frac{d}{2}\big(H_{\frac{d}{2}}-2+\log(4)\big)$	not a function

Table: Basis functions F for different kernels $K(x,y)=F(\|x-y\|)$ and corresponding basis functions f from $\mathrm{k}(x,y)=f(|x-y|)$. Also the inverse Fourier transforms $\mathcal{F}_d^{-1}[F(\|\cdot\|)]$ and $\mathcal{F}_1^{-1}[f(|\cdot|)]$

Applications of the Kernel Summation Problem

- Kernel density estimation [Parzen 1962] [Rosenblatt 1956]
- Classification via support vector machines [Steinwart Christmann 2008]
- Dimensionality reduction with kernelized principal component analysis [Schölkopf Smola 2002] [Shawe-Taylor Cristianini 2004]
- Dithering and halftoning of images [Teuber Steidl Gwosdek Schmaltz Weickert 2010] [Krahmer Veselovska 2024]
- Electrostatic particle simulation [Nestler Pippig Potts 2015]
- Maximum mean discrepancies or the energy distance on the space of probability measure [Gretton Borgwardt Rasch Schölkopf Smola 2006] [Székely 2002]
- Corresponding gradient flows [Arbel Korba Salim Gretton 2019] [Galashov Bortoli Gretton 2024] [Hagemann Hertrich Altekrüger Beinert Chemseddine Steidl 2024] [Kolouri Nadjahi Shahrampour Simsekli 2022]
- Methods for Bayesian inference like Stein variational gradient descent [Liu Wang 2016]

Random Fourier features (RFF)

A continuous, bounded function $G\colon \mathbb{R}^d \to \mathbb{R}$ is called **positive definite** if for all $N \in \mathbb{N}$

$$\sum_{j,k=1}^{N} a_j a_k G(x_j - x_k) \geq 0 \qquad \forall x_j \in \mathbb{R}^d,\ a_j \in \mathbb{R}.$$

$\Leftrightarrow$ Matrix $[G(x_j - x_k)]_{j,k}$ is positive semi-definite

Theorem [Bochner 1933]

Denote by $\mathcal{P}(\mathbb{R}^d)$ the space of probability measures. The **Fourier transform**

$$\mathcal{F}_d\colon \mathcal{P}(\mathbb{R}^d) \to C_0(\mathbb{R}^d), \quad \mathcal{F}_d[\mu](x) = \int_{\mathbb{R}^d} \mathrm{e}^{-2\pi \mathrm{i}\langle x,v\rangle}\, \mathrm{d}\mu(v) = \mathbb{E}_{v\sim\mu}[\mathrm{e}^{-2\pi \mathrm{i}\langle x,v\rangle}],$$

is bijective from $\mathcal{P}(\mathbb{R}^d)$ to the positive definite functions on $\mathbb{R}^d$ with $G(0) = 1$.

If μ has density $g_\mu \in L^1(\mathbb{R}^d)$, then

$$\mathcal{F}_d[\mu](x) = \int_{\mathbb{R}^d} \mathrm{e}^{-2\pi \mathrm{i}\langle x,v\rangle} g_\mu(v)\, \mathrm{d}v.$$

Kernel	positive definite
Gaussian	☑
Matérn	☑
Laplacian	☑
Riesz	✖

Random Fourier features (RFF)

[Rahimi Recht 2007]

- Assumption: $F \circ \|\cdot\|$ positive definite on $\mathbb{R}^d$ and $F(0) = 1$
- $\exists$ Fourier transform $\mu \in \mathcal{P}(\mathbb{R}^d)$ with $F(\|x\|) = \mathbb{E}_{v\sim\mu}[\mathrm{e}^{2\pi\mathrm{i}\langle x,y\rangle}]$
- Draw iid samples $v_1, \dots, v_D$ from μ:

$$F(\|x\|) = \mathbb{E}_{v\sim\mu}\left[\mathrm{e}^{-2\pi\mathrm{i}\langle x,v\rangle}\right] \approx \frac{1}{D}\sum_{p=1}^{D} \mathrm{e}^{2\pi\mathrm{i}\langle x,v_p\rangle}$$

- Insert into the sum (2)

$$\begin{aligned} s_m = \sum_{n=1}^{N} w_n F(\|x_n - y_m\|) &\approx \sum_{n=1}^{N} w_n \frac{1}{D}\sum_{p=1}^{D} \mathrm{e}^{2\pi\mathrm{i}\langle x_n - y_m, v_p\rangle} \\ &= \frac{1}{D}\sum_{p=1}^{D} \mathrm{e}^{-2\pi\mathrm{i}\langle y_m, v_p\rangle} \sum_{n=1}^{N} w_n \mathrm{e}^{2\pi\mathrm{i}\langle x_n, v_p\rangle} \end{aligned}$$

☺ Complexity $\rightsquigarrow \mathcal{O}(D(N+M))$

☺ Simple to implement

😐 Error bound $\mathbb{E}[\|F - \tilde{F}\|_\infty] \in \mathcal{O}(1/\sqrt{D})$ [Sutherland Schneider 2015]

☹ Only for positive definite functions $\rightsquigarrow$ Not applicable for Riesz kernel $F(\|x\|) = -\|x\|$

Slicing Error

