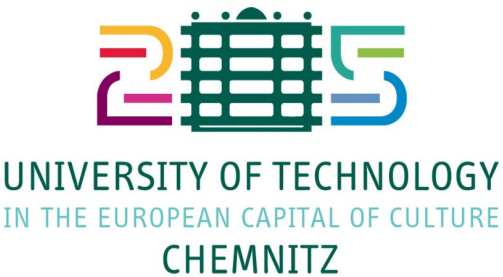
Numerical Methods for Kernel Slicing

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Outline

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Problem Setting

- **Given:** $x_1, \dots, x_N \in \mathbb{R}^d$, $y_1, \dots, y_N \in \mathbb{R}^d$ and $w_1, \dots, w_N \in \mathbb{R}$ and $F\colon [0,1] \to \mathbb{R}$
- **Compute fast**:

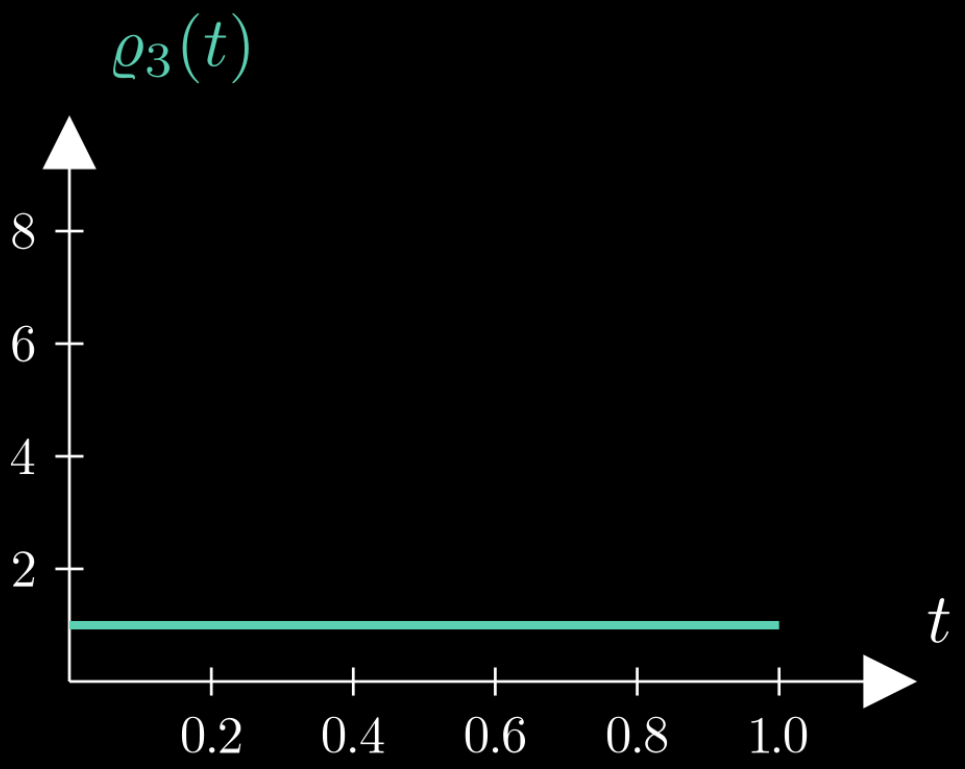
$$s_m := \sum_{n=1}^{N} F(\|x_n - y_m\|_2) w_n \quad m = 1, \dots, N.$$

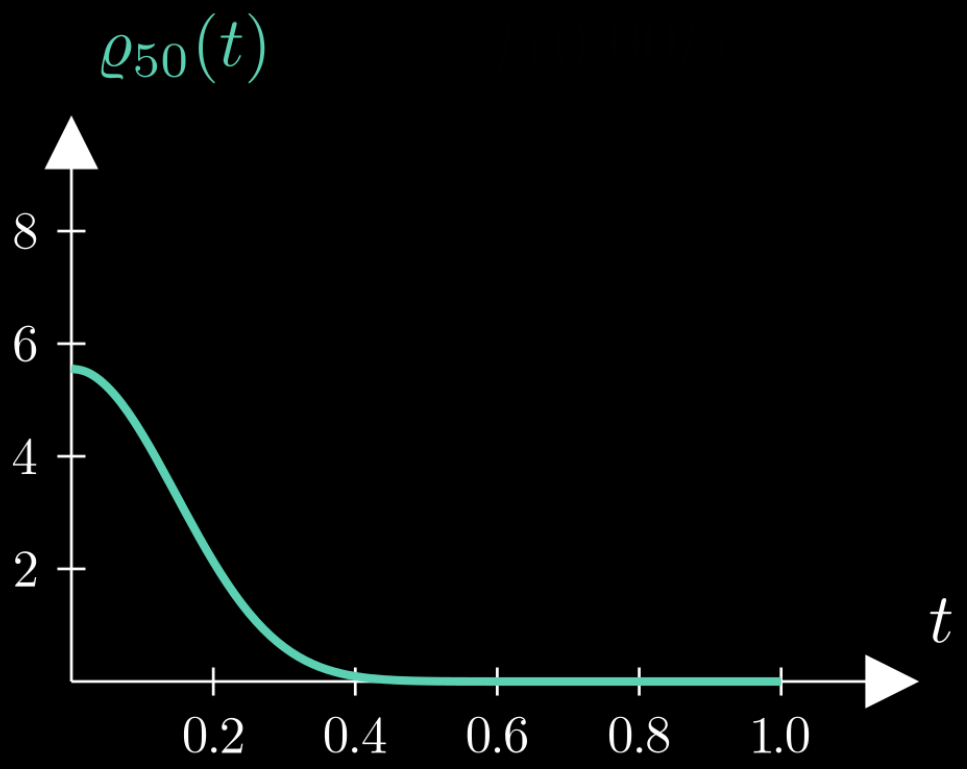
- Fast summation requires Fourier coefficients of $f\colon [0,1] \to \mathbb{R}$:

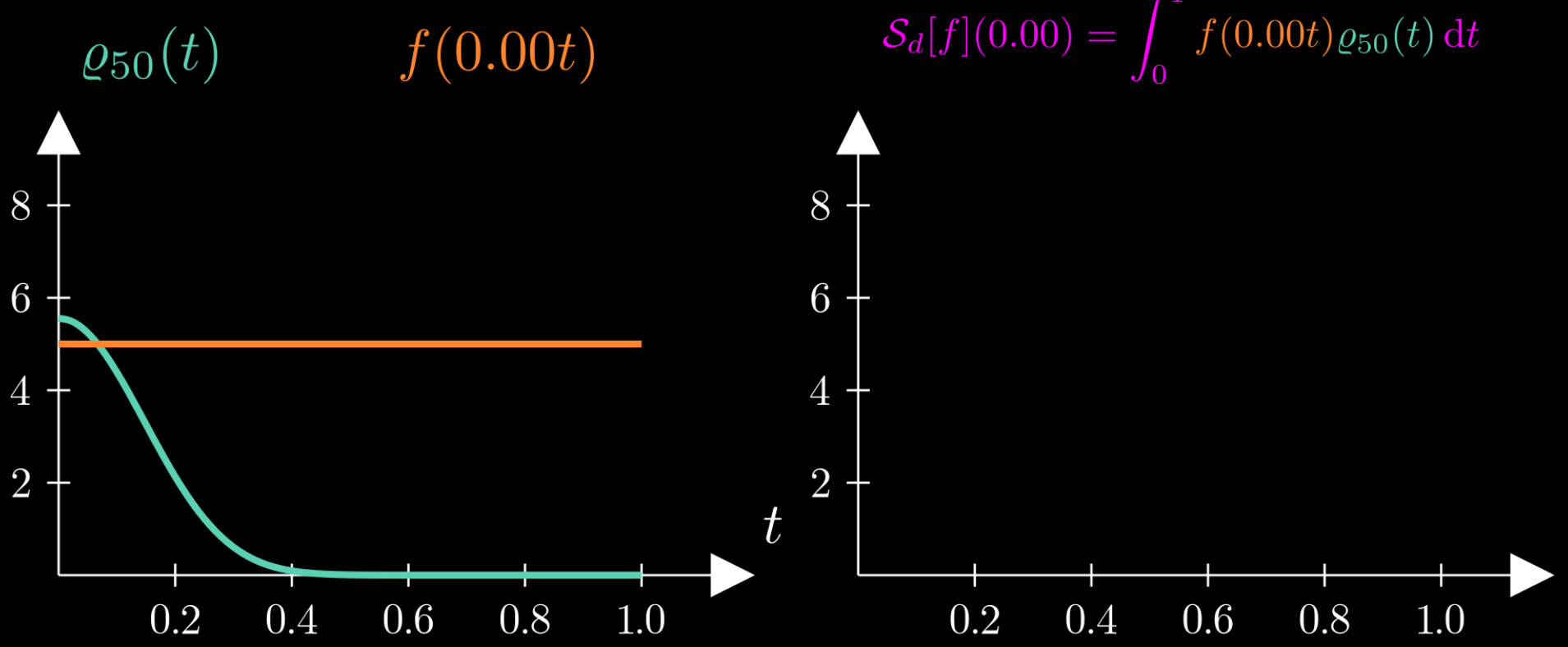
$$F(\|x\|) = \frac{1}{\omega_{d-1}} \int_{\mathbb{S}^{d-1}} f(|\langle x, \xi \rangle|) \mathrm{d}\xi \quad \Leftrightarrow [1] \quad F = \mathcal{S}_d[f]$$

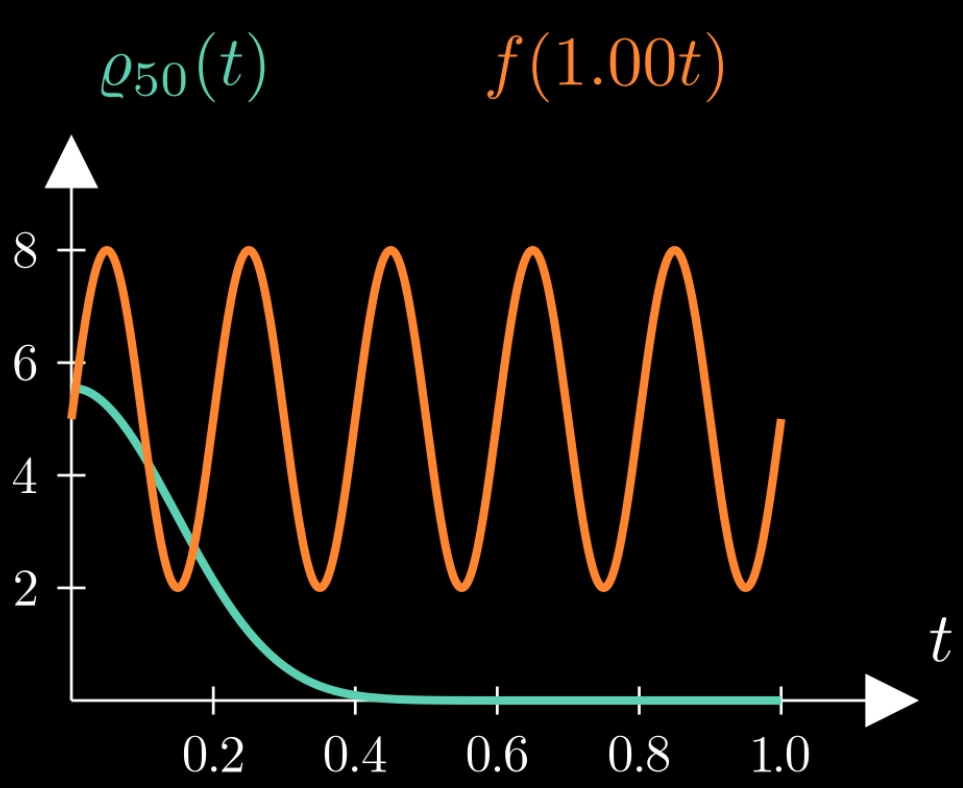
$$\mathcal{S}_d[f](s) = \int_0^1 f(ts)\rho_d(t)\mathrm{d}t \quad \rho_d(t) := c_d(1-t^2)^{\frac{d-3}{2}}$$

[1] [1]

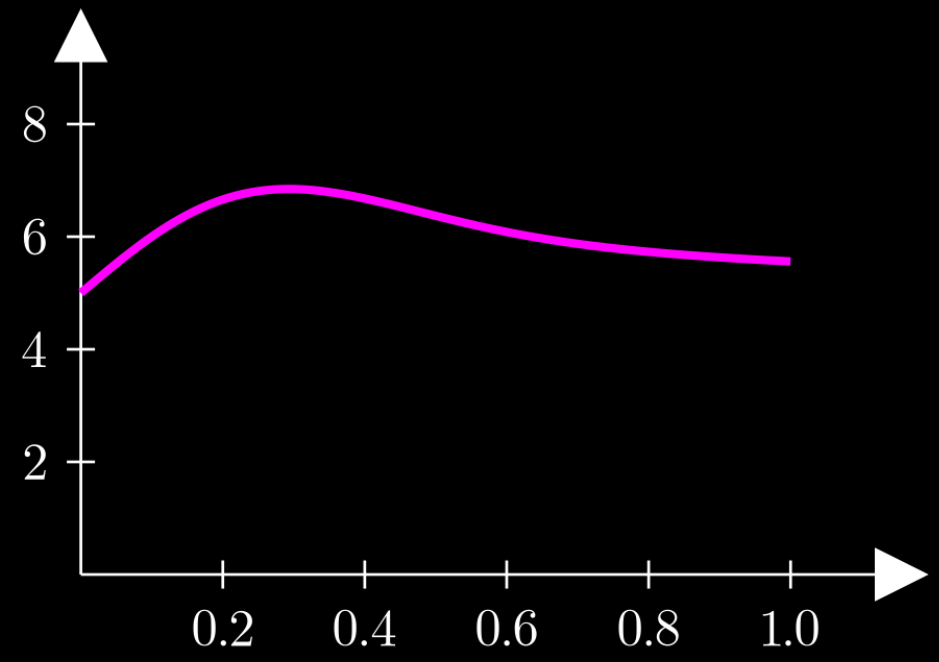

$\varrho_3(t)$
8
6
4
2
t
0.2
0.4
0.6
0.8
1.0


$\varrho_{50}(t)$
8
6
4
2
t
0.2
0.4
0.6
0.8
1.0


$\varrho_{50}(t)$
$f(0.00t)$
$\mathcal{S}_d[f](0.00) = \int_0^1 f(0.00t)\varrho_{50}(t)\,\mathrm{d}t$
8
6
4
2
t
0.2
0.4
0.6
0.8
1.0
8
6
4
2
0.2
0.4
0.6
0.8
1.0



$$\mathcal{S}_d[f](1.00) = \int_0^1 f(1.00t)\varrho_{50}(t)\,\mathrm{d}t$$



An Inverse Problem Perspective

- Cosine orthonormal basis: $g_0(t) := 1$ $g_k(t) := \sqrt{2}\cos(\pi k t)$

$$\hat{a} = \underset{a \in \mathbb{R}^{K+1}}{\arg\min} \|\mathcal{S}_d f_a - F\|_{\mathcal{H}}^2 + \tau^2 \|a\|^2 \quad f_a = a_0 g_0 + \ldots + a_K g_K.$$

- $\tilde{F} = \mathcal{S}_d \tilde{f}$, $1 \leq p \leq \infty$ $\tilde{s}_m := \sum_{n=1}^{N} \tilde{F}(\|x_n - y_m\|_2) w_n$

$$|s_m - \tilde{s}_m| \leq \sum_{n=1}^{N} |F - \tilde{F}|(\|x_n - y_m\|)|w_n| \leq \underbrace{\left\| [(F - \tilde{F})(\|x_n - y_m\|)]_{n=1}^{N} \right\|_p}_{\|F - \tilde{F}\|_{L^p([0,1])}} \|w\|_q$$

- MSE: $p = 2$, worst case bound: $p = \infty$.
- Sobolev embedding $\|F\|_{L^\infty([0,1])} \leq \sqrt{2}\|F\|_{H^1([0,1])}$.
- $\mathcal{H} \in \{L^2([0,1]), H^1([0,1])\}$

Boundedness of the Slicing Operator

Theorem

For $d \geq 3$, the operator $\mathcal{S}_d$ is bounded on the following spaces:

(i) *For $1 < p \leq \infty$, $\mathcal{S}_d \colon L^p([0,1]) \to L^p([0,1])$ is bounded with $\|\mathcal{S}_d\|_{\mathrm{op}} \leq \frac{c_d p}{p-1}$ ($\|\mathcal{S}_d\|_{\mathrm{op}} = 1$ for $p = \infty$).*

(ii) *$\mathcal{S}_d \colon H^1([0,1]) \to H^1([0,1])$, bounded with $\|\mathcal{S}_d\|_{\mathrm{op}} \leq (2 + \frac{c_d}{2})^{\frac{1}{2}}$.*

- Proof by Hardy's inequality [2]: $\|H\|_{L^p \to L^p} = \frac{p}{1-p}$

  $Hf(s) := \frac{1}{s}\int_0^s f(t)\mathrm{d}t = \int_0^1 f(ts)\mathrm{d}t = \mathcal{S}_3 f(s).$
- Statement (i) fails for $p = 1$: $f(t) = t^{-1}\ln(\frac{\mathrm{e}}{t})^{-2}$

Optimization in the Spacial Domain

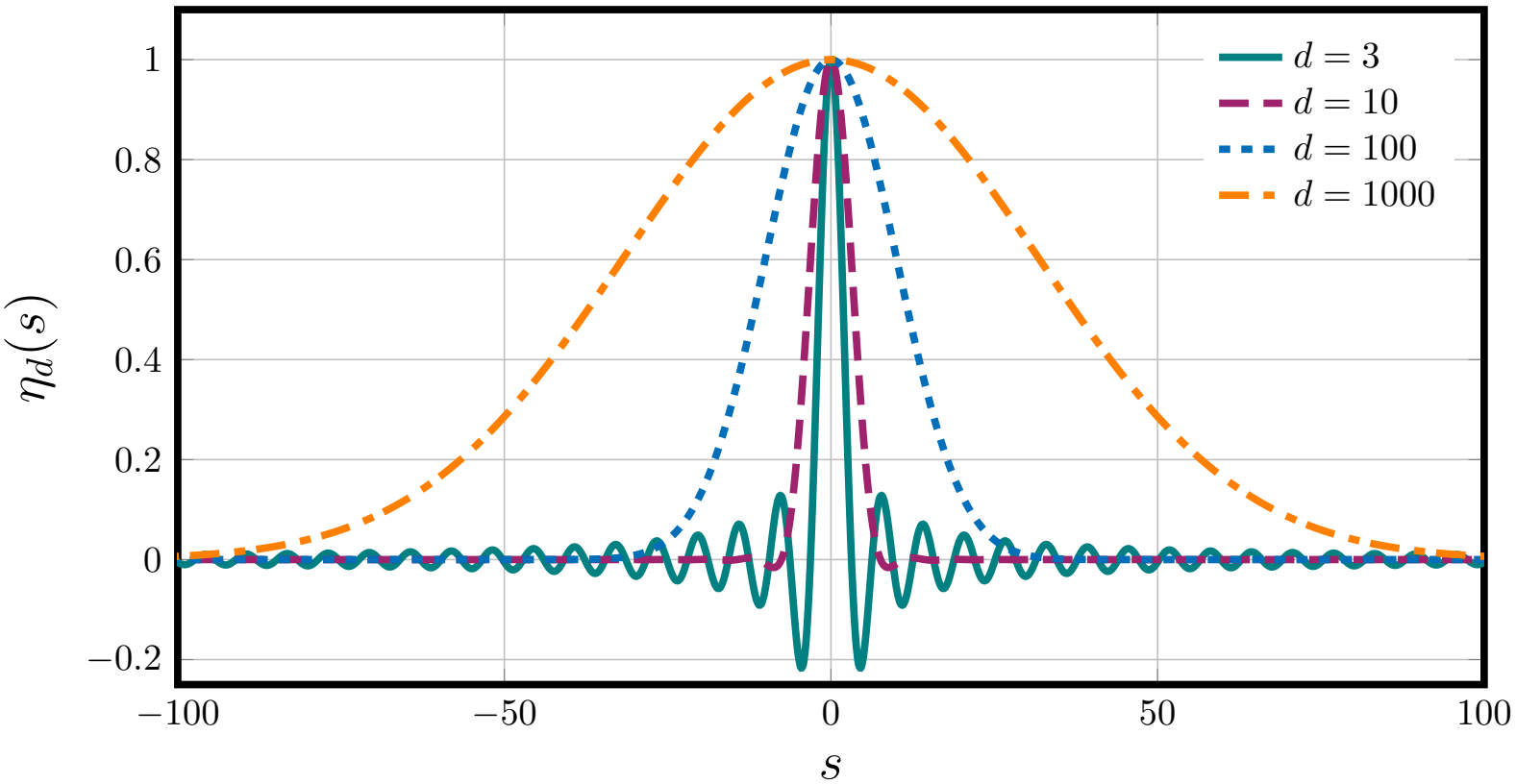
$$\hat{a} = \operatorname*{arg\,min}_{a \in \mathbb{R}^{K+1}} \|\mathcal{S}_d f_a - F\|^2_{L^2([0,1])} + \tau^2 \|a\|^2_{\mathcal{G}} \quad f_a = a_0 g_0 + \ldots + a_K g_K.$$

- For quadrature nodes (x_l, v_l) $l = 1, \ldots, L$ set $H_{k,l} := \mathcal{S}_d[g_k](x_l)\sqrt{v_l}$ and $b_l := F(x_l)\sqrt{v_l}$.

$$\|\mathcal{S}_d f_a - F\|^2_{L^2([0,1])} = \|a_0 \mathcal{S}_d g_0 + \ldots + a_K \mathcal{S}_d g_K - F\|^2_{L^2([0,1])}$$
$$\approx \sum_{l=1}^{L} \left(\sum_{j=1}^{K} \mathcal{S}_d[g_k](x_l) a_k - F(x_l) \right)^2 v_l = \|H^\top a - b\|_2^2$$

- Solve $\operatorname*{arg\,min}_{a \in \mathbb{R}^{K+1}} \|H^\top a - b\|_2^2 + \tau^2 \|Da\|_2^2$
- $\mathcal{S}_d g_k(s) = \sqrt{2}\mathcal{S}_d[\cos](\pi k s) = \sqrt{2}\eta_d(\pi k s)$ with $\eta_d = \mathcal{S}_d[\cos]$.
- $\eta_d = \Gamma(\frac{d}{2})(\frac{2}{s})^{\frac{d}{2}-1} J_{\frac{d}{2}-1}(s) = {}_0F_1(\frac{d}{2}, -\frac{s^2}{4}) = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\frac{d}{2})}{\Gamma(k+1)\Gamma(k+\frac{d}{2})} (\frac{s}{2})^{2k}$

$$\eta_d(s) = \mathcal{S}_d[\cos](s) = \int_0^1 \cos(st)\varrho_d(t)\mathrm{d}t = \frac{1}{2}\mathcal{F}[\varrho_d](s).$$



Lifting into the Frequency Domain

- Lift $\mathcal{H}$ into Fourier space $\mathcal{C}\colon \mathcal{H} \longrightarrow \mathcal{C}(\mathcal{H})$, via

$$\langle a, b\rangle_{\ell(\mathcal{H})} := \langle \mathcal{C}^{-1}a, \mathcal{C}^{-1}b\rangle_{\mathcal{H}} \quad a, b \in \mathcal{C}(\mathcal{H}).$$

- $\|a\|^2_{\ell(L^2([0,1])} = \sum_{k=0}^{\infty} a_k^2$ and $\|a\|^2_{\ell(H^1([0,1])} = \sum_{k=0}^{\infty}(1+\pi^2k^2)a_k^2$
- In the Fourier Space

$$\hat{a} = \operatorname*{arg\,min}_{a\in\mathbb{R}^{K+1}} \|\mathcal{S}_d f_a - F\|^2_{\mathcal{H}} = \operatorname*{arg\,min}_{a\in\mathbb{R}^{K+1}} \|\mathcal{C}\mathcal{S}_d\mathcal{C}^{-1}a - \mathcal{C}F\|^2_{\ell(\mathcal{H})}$$

$$\begin{array}{ccc} L^2([0,1]) & \xrightarrow{\mathcal{S}_d} & L^2([0,1]) \\ \downarrow{\scriptstyle \mathcal{C}} & & \downarrow{\scriptstyle \mathcal{C}} \\ \ell^2 & \xrightarrow{\quad S \quad} & \ell^2 \end{array}$$

$$S_{j,k} = \mathcal{S}_d[\operatorname{sinc}(\cdot + j) + \operatorname{sinc}(\cdot - j)](k)$$
$$k, j \geq 1$$
$$Sa = b \text{ where } b_k = \mathcal{C}[F]_k$$

Implementation Details

Algorithm Frequency & Spacial Domain

Require: $F\colon [0,1] \to \mathbb{R}$, dimension $d \in \mathbb{N}$

Require: Number of cosine coefficients K in domain and J in range, number of Quadrature points L, regularization strength $\tau > 0$

1: **Get** Quadrature points (x_l, v_l) $l = 1, \dots, L$.

2: **Build** $X_{k,j} = \mathcal{S}_d[\operatorname{sinc}(\cdot + k) + \operatorname{sinc}(\cdot - k)](j)$ or $X_{k,j} = \mathcal{S}_d[g_k](x_l)\sqrt{v_l}$.

3: **Compute** $b_k = \mathcal{C}F_k$ or $b_k := F(x_l)\sqrt{v_l}$

4: **Solve** $\hat{a} = \arg\min_a \|Xa - b\|_{\mathcal{H}}^2 + \tau^2 \|Da\|_2^2$

Two Types of Errors

- **Approximation Error**

$$\mathcal{E}_{d,P}[f, F](\|x\|)^2 := \mathbb{E}_{\xi_1,\dots,\xi_P \sim \mathcal{U}_{\mathbb{S}^{d-1}}} \left[\left(\frac{1}{P} \sum_{p=1}^{P} f(|\langle \xi_p, x \rangle|) - F(\|x\|) \right)^2 \right]$$

- **Slicing Error** $\mathcal{E}_{d,P}[f, \mathcal{S}_d[f]](\|x\|)^2 = \frac{\mathbb{V}_d[f](\|x\|)}{P}$ [3]
- **Forward Error** $|\mathcal{S}_d \tilde{f}(\|x\|) - F(\|x\|)|$

Theorem

For any $\tilde{f}, F \in \mathcal{C}([0,1])$ *and* $s \in [0,1]$ *it holds*

(i) $\mathcal{E}_{d,P}[\tilde{f}, F](s)^2 = |F(s) - \mathcal{S}_d[\tilde{f}](s)|^2 + \mathbb{V}_d[\tilde{f}](s) P^{-1}$

(ii) $\|\mathbb{V}_d[f_a]\|_{L^\infty([0,1])} \leq 3 \|f_a\|^2_{H^1([0,1])}$

Noise-free but Regularized

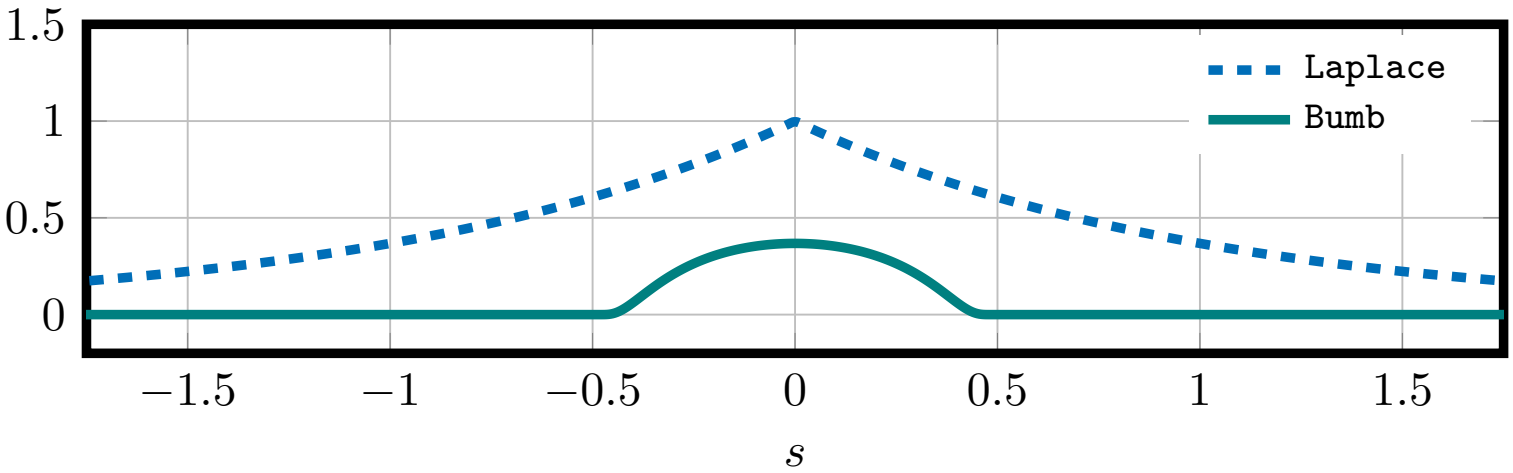
$$\|\mathcal{E}_{d,P}[\mathcal{C}^{-1}a, F]\|^2_{H^1([0,1])} \leq \|\mathcal{S}_d\mathcal{C}^{-1}a - F\|^2_{H^1([0,1])} + \frac{3}{P}\|a\|^2_{\ell(H^1([0,1]))}$$

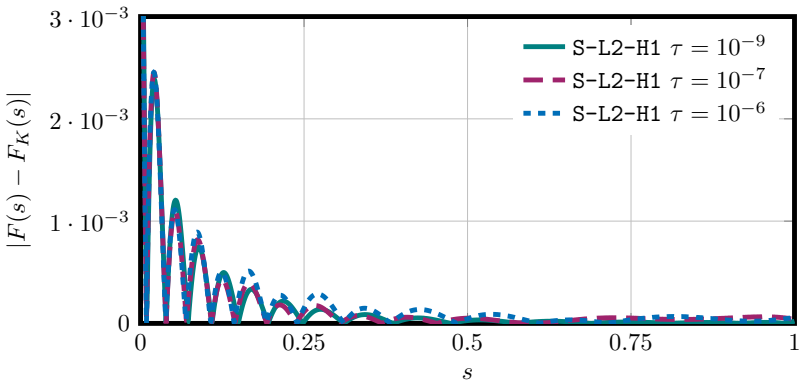
$$\arg\min_{a\in\mathbb{R}^{K+1}} \|Sa - CF\|^2_{\ell(H^1([0,1]))} + \tau^2\|a\|^2_{\ell(H^1([0,1]))}$$

Method	Algorithm	Range $\mathcal{H}$	Domain $\mathcal{G}$
S-L2-H1	Spatial Domain	$L^2([0,1])$	$H^1([0,1])$
F-L2-H1	Frequency Domain	$L^2([0,1])$	$H^1([0,1])$
F-H1-H1	Frequency Domain	$H^1([0,1])$	$H^1([0,1])$

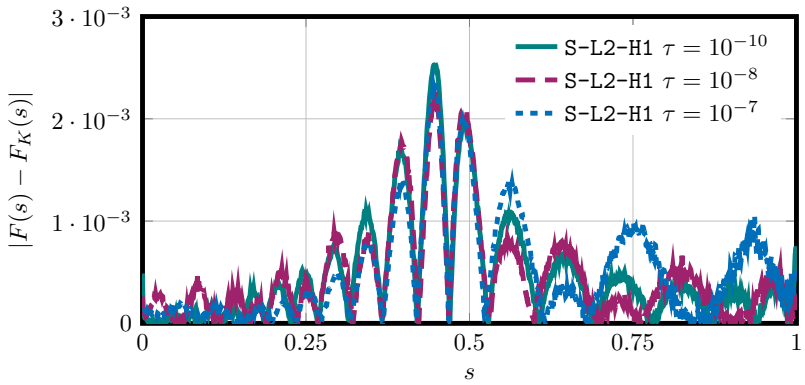
Table: Abbreviations and setup for our numerical tests.

Numerical Results: The Forward Error





(a) The Laplace: $F(s) = \exp(-|s|)$



(b) Bumb: $F(s) = \exp(\frac{1}{4s^2-1})$

Figure: Dimension: $d = 1000$, Coefficients $K = 2^8$, L^{10} quadrature nodes

Thank you for your attention :)

I am happy to take any questions!

References I

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- G. H. Hardy, "Note on a theorem of Hilbert," *Mathematische Zeitschrift*, vol. 6, no. 3, pp. 314–317, 1920.
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- N. Rux, M. Quellmalz, and G. Steidl, "Slicing of radial functions: a dimension walk in the Fourier space," *Sampling Theory, Signal Processing, and Data Analysis*, vol. 23, no. 1, pp. 1–40, 2025.

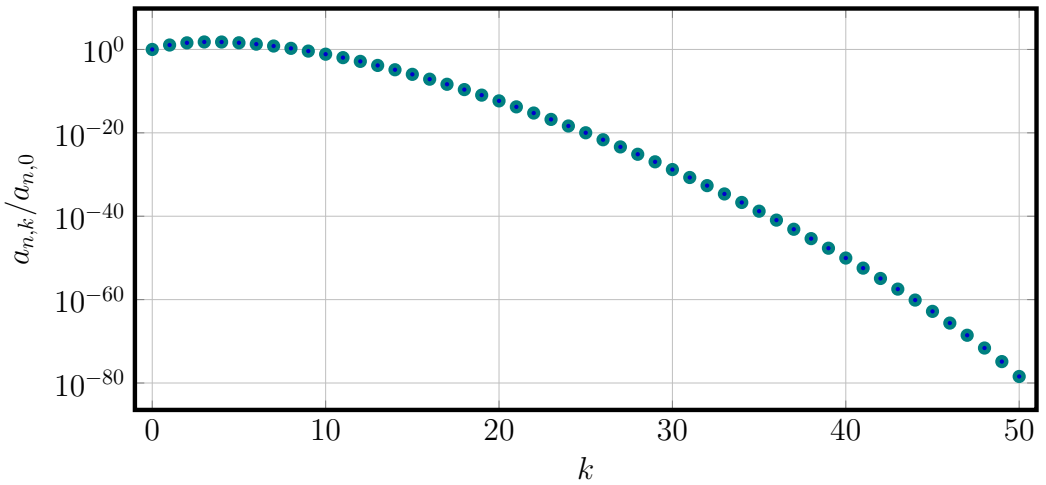
The Inverse Slicing Operator

Let $d \geq 3$ be odd and set $n := \frac{(d-1)}{2}$, $\mathcal{S}_d^{-1} : H^n([0,1]) \to L^2([0,1])$

$$\mathcal{S}_d^{-1}[F](t) = \frac{2^n n!}{(2n)!} \sum_{k=0}^{n} a_{n,k} t^k F^{(k)}(t)$$

where $a_{m,m} = 1$ and $a_{m,k} \in \mathbb{N}$ recursively for $k = 1, \ldots, m$ through

$$a_{m,0} = (d-2m)a_{m-1,0} \text{ and } a_{m,k} = (d-2m+k)a_{m-1,k} + a_{m,k-1}.$$



$d = 101$